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On Minimal Non PT-Groups

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Abstract

A finite group G is said to be a PT-group if every subnormal subgroup of G is permutable in G. In this paper, we determine the structure of a non PT-group all of its proper subgroups are PT-groups.

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1 Introduction

All groups considered in this paper will be finite. A group G is said to be a T-group if every subnormal subgroup of G is normal in G. Such groups were introduced by Gaschütz [6]. A subgroup H of a group G is said to be permutable (or quasinormal)in G if it permutes with every subgroup of G. A group G is said to be a PT-group if every subnormal subgroup of G is permutable in G. Solvable PT-groups were studied and classified by Zacher [11]. Clearly, every T-group is a PT-group. The converse is not true. For example, any non-dedekind modular p-group is a PT-group but not a T-group. Following Beidleman et al. [4], a group G satisfies the property X_p, where p is a prime, if and only if each subgroup of a Sylow p-subgroup P of G is permutable in $N_G(P)$. They showed that G is a solvable PT-group if and only if it satisfies X_p for all primes p. A subgroup H of G is said to be S-permutable in G if it permutes with every Sylow subgroup of G. A group G is said to be a PST-group if every subnormal subgroup of G is S-permutable in G. The structure of solvable PST-groups was determined by Agrawal [1]; see also Asaad and Csörgö [3].

If $\mathcal{P}$ is a property of groups, a group G is said to be a minimal non $\mathcal{P}$-group if G does not have $\mathcal{P}$ but all its proper subgroups do. Minimal nonnilpotent groups have been studied in detail by Schmidt, Iwasawa and Golfand; Redei gave the complete classification of such groups (see [8, Satz 5.2, p. 281]). Also, Doerk [5] determined the structure of minimal non-supersolvable groups (non supersolvable groups all of its proper subgroups are supersolvable). Our object here is to determine the structure of minimal non PT-groups (non PT-groups all of its proper subgroups are PT-groups).

Our notation is standard and taken mainly from [9]. In addition, $|\pi(G)|$ will denote the number of distinct prime divisors of $|G|$.

2 Preliminaries

In this section, we collect some of the results that will be used later.

Lemma 2.1 *A group G satisfies X_p if and only if a Sylow p-subgroup P of G is modular and every normal subgroup of P is pronormal in G.*

Proof. See [4, Theorem D].

Lemma 2.2 *Let G be a group satisfies X_p and let P be a Sylow p-subgroup of G. Then either G is p-nilpotent or P is abelian.*

Proof. See the proof of theorem B of [4].

Lemma 2.3 *If H is a normal Hall subgroup of a group G such that G/H is a PT-group and all subnormal subgroups of H are normal in G, then G is a PT-group.*

Proof. See [4, Lemma 1].

Lemma 2.4 *A group G is a solvable PT-group if and only if it satisfies X_p for all primes p.*

Proof. See [4, Theorem A].

Lemma 2.5 *Let M be a normal p'-subgroup of G. Then G satisfies X_p if and only if G/M satisfies X_p.*

Proof. See [2, Lemma 4.1].

Lemma 2.6 *Let G be a solvable PT-group. Then*
(i) Factor groups of G are solvable PT-groups.
(ii) G is supersolvable.

Proof.

(i) Let $H \leq K \leq G$ and $H \triangleleft G$. Suppose that K/H is a subnormal subgroup of G/H. Obviously, K is subnormal in G and then K is permutable in G. Hence $KL = LK$ for every subgroup L of G and so $(K/H)(LH/H) = (LH/H)(K/H)$ for every subgroup LH/H of G/H. Thus K/H is permutable in G/H as desired.

(ii) We shall distinguish two possible situations depending on $Z(G) = 1$ or not. If $Z(G) \neq 1$, then $G/Z(G)$ is a solvable PT-group by (i). Thus $G/Z(G)$ is supersolvable by induction on $|G|$ and so G is supersolvable.

On the other hand if $Z(G) = 1$, then G is a solvable T-group by [4, Corollary 1(4)] and so G is supersolvable, an observation due to Gaschütz [6].■

The converse of statement of Lemma 2.6 (ii) is not true. This can be easily seen by examining of the dihedral group of order 8.

Lemma 2.7 *A group G is a subgroup closed PT-group if and only if G is a solvable PT-group.*

Proof. See Zacher [11], and the rest follows from the fact that a group whose maximal subgroups are supersolvable is solvable. ■

3 Main results

We need the following lemmas:

Lemma 3.1 *If a group G possesses three solvable PT-subgroups whose indices are pairwise relatively prime, then G is a solvable PT-group.*

Proof. We prove the result by induction on $|G|$. It is known that G is solvable [9, Exercise 9.1.9]. Let H_i ,$1 \leq i \leq 3$, be the three given subgroups of G. If $H_1 = 1$, then $|G : H_1| = |G|$. Then $|G : H_2|$ must be relatively prime to $|G|$, which is possible only if $H_2 = G$, whence G is a solvable PT-group in this case. Hence we may assume that $H_i \neq 1$, $1 \leq i \leq 3$. Let M_i, $1 \leq i \leq 3$ be three maximal subgroups of G such that $H_i \leq M_i$. Then $M_1 = H_1(H_2 \cap M_1)$ and $M_1 = H_1(H_3 \cap M_1)$. Thus M_1 possesses three solvable PT-subgroups $H_1, H_2 \cap M_1, H_3 \cap M_1$ whose indices are pairwise relatively prime. Then M_1 is a solvable PT-group by induction on $|G|$. Similarly, M_2 and M_3 are solvable PT- groups. Let N be an arbitrary maximal subgroup of G. Hence if N is a conjugate to M_i for some i, , $1 \leq i \leq 3, N$ is a solvable PT-group. Thus we may assume that N is not conjugate to M_i, $1 \leq i \leq 3$. Then $G = NM_i$, $1 \leq i \leq 3$ and so $|G : M_i| = |N : N \cap M_i|$, $1 \leq i \leq 3$. Now, we conclude that N possesses three solvable PT-subgroups $(N \cap M_i)$, $1 \leq i \leq 3$

whose indices are pairwise relatively prime. Hence N is a solvable PT-group by induction on $|G|$. Since N is an arbitrary maximal subgroup of G, it follows that all maximal subgroups of G are solvable PT-groups. Hence, all proper subgroups of G are solvable PT-groups.

If G is a PT-group, we are ready. Thus we assume that G is not a PT-group. If G is a solvable PST-group, then G is supersolvable by [3, Lemma 1(ii)] and so there exists a normal Sylow p-subgroup P for some prime p in $\pi(G)$. Also, if we assume that G is not a PST-group and since all proper subgroups of G are PT-groups, whence all proper subgroups are PST-group, it follows by [3, Corollary 5] that there exists a normal Sylow p-subgroup P for some prime p in $\pi(G)$. By Schur- Zassenhause's Theorem, $G = PK$ where K is a p'-Hall subgroup of G. We argue that G satisfies X_p for all primes p. Let L be a normal subgroup of P. Since $|\pi(G)| \geq 3$, we have that PQ is a PT-group where Q is a Sylow q-subgroup of G, $q \neq p$. Since L is subnormal in PQ, it follows that L is permutable in PQ and hence LQ is a subgroup of PQ. Clearly, L is normalized by Q and so L is normalized by $O^p(G)$, where $O^p(G) =< Q : Q$ is a Sylow q-subgroup of $G, q \neq p >$. Since $L \lhd P$, we have that $L \lhd G$. Note that P is modular. Thus by Lemma 2.1, G satisfies X_p. So Lemma 2.2 implies that P is abelian or G is p-nilpotent. If P is abelian, it is easy to show that every subgroup of P is normal in G and so by Lemma 2.3, we have that G is a PT-group, a contradiction. Thus we may assume that G is p-nilpotent. Then there exists a normal p'-Hall subgroup K of G such that $G = P \times K$. Thus $G/P \cong K$ is a PT-group and so by Lemma 2.5, $G/P \cong K$ satisfies X_q for all primes q divides $|K|$. Note that P is a normal q'-subgroup of G, where $q \neq p$. Thus by Lemma 2.5, G satisfies X_q for all primes $q \neq p$. Thus by Lemma 2.5, G satisfies X_q for all primes $q \neq p$. Since G satisfies X_p, G satisfies X_p for all primes p. Applying Lemma 2.4, we have that G is a PT-group; a final contradiction. Thus G is a solvable PT-group.■

Lemma 3.2 *Let G be a solvable group. If G is a minimal non PT-group, then* $1 \leq |\pi(G)| \leq 2$.

Proof. Suppose the result is false and let G be a counterexample of minimal order. Then $|\pi(G)| \geq 3$. So there will be three proper subgroups H_1, H_2 and H_3 are pairwise relatively prime indices which are PT-groups. By Lemma 3.1, G is a PT-group, A contradiction. Thus $1 \leq |\pi(G)| \leq 2$.■

Lemma 3.3 *If G is a minimal non PT-group, then* $1 \leq |\pi(G)| \leq 2$.

Proof. Let p be the smallest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Suppose that G does not have a p-nilpotent. By [7, Theorem 14.4.7, p. 217], there is a subgroup P_1 of P which is normalized but it is not centralized by an element x of order prime to p. Then $P_1 < x >$ is solvable.

If $P_1 < x >= G$, we are ready. Thus we may assume that $P_1 < x >< G$ and so $P_1 < x >$ is a solvable PT-group. So, $P_1 < x >$ is supersolvable by Lemma 2.6 and hence $P_1 < x >= P_1 \times < x >$ and this is impossible. Thus G is p-nilpotent. Hence $G = PK$, where K is a p'-Hall subgroup of G. So K is a PT-group and all subgroups of K are PT-groups. Thus Lemma 2.7 implies that K is solvable whence G is solvable. Now by Lemma 3.2, we have that $1 \leq |\pi(G)| \leq 2$.■

We can now prove:

Theorem 3.4 *If G is a minimal non PT-group, then one of the following statements holds:*

(i) $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G for some distinct primes p and q; or

(ii) G is a p-group for some prime p in which it is either the dihedral group of order 8 or non abelian group of order p^3 of exponent p for $p > 2$.

Proof. By Lemma 3.3, $1 \leq |\pi(G)| \leq 2$. If $|\pi(G)| = 2$, then $G = PG$ with a Sylow p-subgroup P and a Sylow q-subgroup Q for two primes $p \neq q$. By Burnside's Theorem, G is solvable and so all proper subgroups are supersolveble. Then there exists a normal Sylow p-subgroup P for some prime p in $\pi(G)$, say. If Q is normal in G, then $G = P \times Q$. Applying Lemma 2.5, we conclude that G satisfies X_p and X_q. Thus by Lemma 2.4, G is a solvable PT-group; a contradiction. Hence Q is not normal in G.

Now we show that $G/P \cong Q$ is cyclic.

Let $Q_1 < Q$ be a maximal subgroup of Q. Then PQ_1 is a PT-group by hypothesis and it is easy to show that the subgroups of P are normalized by Q_1. If Q is not cyclic, then it has two distinct maximal subgroups Q_1 and Q_2. It follows that Q normalizes all subgroups of P. Clearly, every maximal subgroup of P is normal in G. We will see that every subgroup of P is permutable in G. Let $L \leq P$. We assume that $L < P$. Let M be a maximal subgroup of P such that $L \leq M$. Take $T = MQ$. Since $M \neq P$, it is clear that $M \neq G$ and so $T \neq G$. Thus L is a permutable subgroup of T. Through this agreement replacing Q with some of its conjugate if necessary, we have that L permutes with every q-subgroup of G. But since P is a modular group, it follows that L permutes with every p-subgroup of G. We conclude the proof just by expressing any other subgroup of G as a product of a p-subgroup and a q-subgroup to show that L is permutable in G. Hence G satisfies X_p. Also, since P is a normal q'-subgroup of G, it follows by Lemma 2.5 that G satisfies X_q. Therefore by Lemma 2.4, G is a PT-group; a contradiction. Thus Q is cyclic and so (i) holds.

If $|\pi(G)| = 1$, then G is a p-group for some prime p. Since G is not a PT-group, we have that G is not an M-group. By [10, Lemma 2.3.3], we have that there exist subgroups H and K of G with $K \lhd H$ such that H/K is dihedral group of order 8 or nonabelian of order p^3 of exponent p for $p > 2$. Since all proper subgroups of G are PT-groups, we conclude that G is either the dihedral group of order 8 or nonabelian group of order p^3 of exponent p for $p > 2$. Thus (ii) holds.■

Proposition 3.5 *Let G be a p-group. If all its subgroups are PT-groups, then one of the following statements holds:*

(i) $G = Q_8 \times S$, where Q_8 is a quaternion group of order 8 and S is an elementary abelian 2-group; or

(ii) G contains an abelian normal subgroup A with cyclic factor group G/A; further there exists an element $b \in G$ with $G = A < b >$ and a positive integer s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$.

Proof. By [10, Lemma 2.3.2], we have that G has a modular subgroup lattice and by [10, Theorem 2.3.1], we conclude the result.■

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